Unit - 3

Sequence :

Def^n : If for every positive integer 'n' there is associated a unique real number s_n, then ordered set of numbers $s_1, s_2, s_3, \cdots s_n \cdots$ or $(s_1, s_2, s_3 \cdots s_n \cdots)$ is called sequence. s_n is called n^{th} term or general term of sequence.

The sequence is written as $\{s_n\}_{n=1}^{\infty}$ or $\{s_n\}$

In other word

A sequence is a function whose domain is the set N of all natural numbers where as the range may be any set S, (a subset of the set R of real numbers).

i.e. $f : N \to S$ where $S \subseteq R$

OR $f : N \to R$

Range of a sequence: The set of distinct term of the sequence is called its range.

For example : ① $\{x_n\} = \{(-1)^n\}_{n=1}^{n=\infty}$ be any sequence ~~then~~ i.e.

$\{x_n\} = \{-1, 1, -1, 1, -1, \cdots\}$

The range of this sequence $\{x_n\} = \{-1, +1\}$ finite

(ii) $\{x_n\} = \{(2)^n\} = \{1, 2, 4, 8, \cdots 2^n \cdots\}$

The range of this sequence $\{x_n\} = \{1, 2, 4, 8, \cdots\}$ infinite

(iii) $\{x_n\} = \{(\frac{1}{n})\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \cdots \frac{1}{n}, \cdots\}$

The range of this sequence $\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \cdots\}$ infinite

Note :- If the domain of a sequence is finite set, Then the sequence is called finite sequence. i.e. If the sequence has a finite number of terms called finite sequence.

Bounded and Unbounded Sequence:

A sequence $\{a_n\}$ is said to be bounded sequence ~~when~~ ~~is~~ if there exists two real numbers k and K $(k \leq K)$ such that $k < a_n < K$ for all $n \in N$

A sequence $\{a_n\}$ is said to be bounded above if $\exists$ a real number K such that $a_n \leq K \quad \forall n \in N$

A sequence $\{a_n\}$ is said to be bounded below if $\exists$ a real no k such that $a_n \geq k \quad \forall n \in N$

In other word

A sequence $\{a_n\}$ is said to be bounded when it is bounded above and bounded below;

For example: (I) $\{a_n\} = \{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \frac{1}{n}, \cdots\}$

It is clear that $0 < a_n \leq 1 \quad \forall n \in N$

Hence $\{a_n\}$ is bounded sequence.

(II) A sequence $\{a_n\} = \{(2)^{n-1}\} = \{1, 2, 2^2, 2^3, \cdots 2^n, \cdots\}$

clearly $a_n \geq 1 \quad \forall n$ and $a_n \nless K$ (any real number) i.e

$$1 \leq a_n \nless K$$

Thus sequence $\{a_n\}$ is unbounded (Not bounded)

Limit of a sequence:

Let $\{a_n\}$ be the sequence of real numbers. A real number l is said to be the limit of the sequence if for every given $\varepsilon > 0$ there exists a +ve integer N (depending on ε) such that

$$|a_n - l| < \varepsilon \quad \forall \quad n \geq N$$

We write $\lim_{n\to\infty} a_n = l$ OR $a_n \to l$ as $n \to \infty$

If the limit of a sequence exists then it is always unique.

Convergent, Divergent and Oscillating Sequences:

Convergent sequence: A sequence $\{a_n\}$ is said to be convergent sequence if $\lim_{n\to\infty} a_n = l$ (finite real number)

For example: $\{a_n\} = \{(\frac{1}{2})^n\} = \{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} \cdots \frac{1}{2^n} \cdots\}$

Here $a_n = \frac{1}{2^n}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$ (finite)

Hence this sequence $\{a_n\}$ is convergent. and, this sequence $\{a_n\}$ converges to 0 (zero).

Divergent sequence: A sequence $\{a_n\}$ is said to be divergent if $\lim_{n\to\infty} a_n$ is not finite i.e.

$\lim_{n\to\infty} a_n = +\infty$ or $-\infty$

For example: $\{a_n\} = \{n^2\} = \{1^2, 2^2, 3^2, 4^2 \cdots n^2 \cdots\}$

Here $a_n = n^2$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = +\infty$

This Sequence $\{n^2\}$ is divergent sequence.

Oscillatory Sequence: If a sequence $\{a_n\}$ is neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called oscillatory sequence.

For example: A sequence $\{a_n\} = \{(-1)^n\}$, Here $a_n = (-1)^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n = \begin{cases} +1 & \text{when } n \text{ is even} \\ -1 & \text{when } n \text{ is odd} \end{cases}$$

Hence $\lim_{n\to\infty} a_n$ does not exists and oscillates finitely from -1 to $+1$.

$\Rightarrow \{a_n\} = \{(-1)^n\}$ is an oscillatory sequence.

Monotonic Sequences:

A sequence $\{a_n\}$ is said to be monotonic iff it is either monotonically increasing or monotonically decreasing.

A sequence $\{a_n\}$ is said to be monotonically decreasing if $a_{n+1} \le a_n \quad \forall\ n \in N$

i.e. $a_1 \ge a_2 \ge a_3 \ge a_4 \cdots \ge a_n \ge a_{n+1} \ge \cdots$

A sequence $\{a_n\}$ is said to be monotonically increasing if $a_{n+1} \ge a_n \quad \forall\ n \in N$

$a_1 \le a_2 \le a_3 \le a_4 \le \cdots \le a_n \le a_{n+1} \le \cdots$

Note: (I) If $a_{n+1} > a_n \ \forall\ n \in N$ Then sequence $\{a_n\}$ is called strictly monotonically ~~strictly~~ increasing sequence.

(II) If $a_{n+1} < a_n \ \forall\ n \in N$ Then sequence $\{a_n\}$ is called strictly monotonically decreasing sequence.

(III) The necessary and sufficient condition for the convergence of monotonic sequence is that it is bounded.

(IV) If $\{a_n\}$ and $\{b_n\}$ are two convergent sequences such that $\lim_{n\to\infty} a_n = A$ (finite) and $\lim_{n\to\infty} b_n = B$ (finite)

Then

(I) The sequence $\{a_n b_n\}$ is also convergent and converges to AB

(II) The sequence $\left\{\frac{a_n}{b_n}\right\}$ is also convergent and converges to $\frac{A}{B}$ $\{B \ne 0$

(III) The sequence $\{a_n + b_n\}$ is also convergent and converges to $(A+B)$.

(IV) The sequence $\{a_n\} = \{(-1)^n\}$ is bounded by -1 & $+1$ but this sequence is not convergent.

(V) Every convergent sequence is bounded but converse is not necessarily true.

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Series

If (a_n) be any sequence of real numbers then the expression

$$a_1 + a_2 + a_3 + \cdots\cdots + a_n + \cdots\cdots$$

is called an infinite series. This series is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$

NOTE: Partial sum of an infinite series is the sum of first n-terms of the infinite series. It is denoted by S_n

i.e. $S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$

These sums $S_1, S_2, S_3 \cdots S_n$ are called partial sums

Convergent Series: Let $\sum a_n$ be an infinite series of +ve terms then this series is called convergent series if $\lim_{n\to\infty} S_n$ = finite & unique.

Where S_n = partial sum of the series i.e. the sum of first n-terms of the series

For Ex $\sum u_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = \frac{1\,(1 - 1/2^n)}{(1 - 1/2)}$$

$S_n = 2\,(1 - 1/2^n)$. Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\,(1 - 1/2^n) = 2$

$\lim_{n\to\infty} S_n = 2$ finite & unique.

Hence this series is a convergent series.

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Some Properties of Convergent and Divergent Series:

1. The nature of the series remains unchanged when the sign of all its terms are changed

2. The nature of the series remains unchanged by adding or removal of finite number of terms.

3. The nature of series remains unchanged by multiplied ~~add~~ or divided by any non-zero quantity.

4. If Σa_n and Σb_n are any two infinite series and The sum of these two series:

(i) $\Sigma(a_n+b_n)$ is convergent if both the series Σa_n & Σb_n are convergent

(ii) $\Sigma(a_n+b_n)$ is divergent if any one of Σa_n & Σb_n is divergent.

Some important limits to learn:

(i) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (ii) $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$

(iii) $\lim_{n\to\infty}\frac{x^n}{\underline{|n}} = 0$ where x is any number

(iv) $\lim_{n\to\infty}\frac{1}{n}(\log n)^m = 0$ (v) $\lim_{n\to\infty}\left(\frac{\log n}{n}\right) = 0$

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Let us discuss the nature of geometric series:

Let us consider a geometric series

$$\Sigma u_n = 1 + x + x^2 + x^3 + \dots + x^n + \dots$$

To discuss the nature of the above geometric series we have ~~and~~ consider three cases.

Case I when $x < 1$ then the partial sum of the given geometric series is

S_n = The sum of first n-terms of the series.

$$S_n = 1 + x^1 + x^2 + x^3 + \dots + x^{n-1}$$

$$S_n = \frac{1.(1-x^n)}{(1-x)} \qquad \left\{ \begin{array}{l} \text{G.P.} \\ S_n = \frac{a(1-r^n)}{(1-r)}, r<1 \\ \text{where } a \text{ is F.T} \\ r = \text{C.R.} \end{array} \right.$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{(1-x^n)}{(1-x)}$$

$$= \lim_{n\to\infty} \frac{1}{1-x} - \lim_{n\to\infty} \frac{x^n}{(1-x)}$$

Since $x < 1$ Then $x^n \to 0$ as $n \to \infty$

$\lim_{n\to\infty} S_n = \frac{1}{1-x}$ finite ~~unique~~ quantity for $x<1$

Hence in this case $\Sigma u_n = 1 + x + x^2 + x^3 + \dots$ this series is convergent series.

Case II when $x > 1$ in this case the partial sum $S_n = 1 + x + x^2 + \dots + x^{n-1}$

$$S_n = \frac{1(x^n - 1)}{(x-1)} \qquad \left\{ \begin{array}{l} \text{In G.P} \\ S_n = \frac{a(r^n-1)}{1-r}; r>1 \end{array} \right.$$

$$S_n = \frac{x^n - 1}{(x-1)}$$

$$S_n = \cancel{\text{to}}\ \frac{x^n-1}{x-1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{(x^n-1)}{(x-1)} = \lim_{n\to\infty}\left\{\frac{x^n}{(x-1)} - \frac{1}{(x-1)}\right\}$$

Since $x > 1$ Then $x^n \to \infty$ as $n \to \infty$

$$\lim_{n\to\infty} S_n = \infty - \left(\frac{1}{x-1}\right) = \infty$$

Hence in this case the given geometric series $1 + x + x^2 + x^3 + \cdots + x^n$ is divergent series.

Case-III :- When $x = 1$ Then the partial sum is

$$S_n = 1+1+1+\cdots+1 \text{ (upto n-terms)}$$
$$= n.$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(n) = \infty$$

In this case the given geometric series $1+x+x^2+x^3+\cdots+x^n$ is divergent series.

We conclude that

$\sum_{n=0}^{\infty} x^n$ is convergent if $x < 1$
is divergent if $x \geq 1$

Comparision test (Comparison Test)

If Σv_n and Σu_n are two infinite series of positive terms and Σv_n is convergent then Σu_n is convergent if $u_n \leq v_n \ \forall\ n \in N$ and Σu_n is divergent if Σv_n is divergent and $u_n \geq v_n \ \forall\ n \in N$

In other word The comparison test states that

"The infinite series $\sum u_n$ is convergent or divergent according as the series $\sum v_n$ is convergent or divergent if $\lim_{n\to\infty} \frac{u_n}{v_n}$ is finite

Ex. Test the series whose nth term is

$$u_n = \{\sqrt{n^2+1} - n\}$$

Sol^n

$$u_n = \{(n^2+1)^{1/2} - n\}$$ is nth term of the series $\sum u_n$.

Now we shall find Auxiliary series $\sum v_n$.

$$u_n = \{(1+n^2)^{1/2} - n\} \qquad \left[(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \dots \right.$$

$$u_n = \left\{ \cancel{1 + \tfrac{1}{2} n^2 + \frac{\tfrac12(\tfrac12 - 1)}{\lfloor 2}(n^2)^2 + \dots} - n \right\}$$

$$u_n = n\left\{(1 + 1/n^2)^{1/2} - 1\right\}$$

$$u_n = n\left\{\cancel{1} + \frac{1}{2}\cdot\frac{1}{n^2} + \frac{\tfrac12(\tfrac12 - 1)}{\lfloor 2}\cdot\frac{1}{n^4} + \dots - \cancel{1}\right\}$$

$$u_n = \frac{1}{2n} - \frac{1}{8}\frac{1}{n^3} + \dots$$

$$u_n = \frac{1}{n}\left\{\frac{1}{2} - \frac{1}{8}\cdot\frac{1}{n^2} + \dots\right\}$$

Let us denote the auxiliary series $\sum v_n$ whose nth term is $v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left\{\frac{1}{2} - \frac{1}{8}\cdot\frac{1}{n^2} + \dots\right\}$$

$$= \frac{1}{2} \text{ finite}$$

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Since $\sum v_n = \sum \frac{1}{n}$ is divergent series
Hence by comparison test the given series
$\sum u_n$.

D'Alembert's Test or Ratio Test

Statement:- A series $\sum u_n$ of positive terms is convergent if from and after some fixed terms $\frac{u_{n+1}}{u_n} < r < 1$, where r is a fixed number. The series is divergent if $\frac{u_{n+1}}{u_n} > 1$ from and after some fixed terms.

In other word: The series $\sum u_n$ of positive terms is convergent if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$ and divergent if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} < 1$.
if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$ Then D'Alembert's test fails.

Ex Test the convergence of the series.

$$\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots\cdots$$

Sol^n let $\sum u_n = \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots\cdots$

Here nth term of the above series is

$$u_n = \frac{x^n}{n\cdot(n+1)}$$

$$u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{n\cancel{(n+1)}} \times \frac{\cancel{(n+1)}(n+2)}{x^{n+1}} = \left(\frac{n+2}{n}\right)\cdot\frac{1}{x}$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{n+2}{n}\right)\cdot\frac{1}{x}$$

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$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)\cdot\frac{1}{x}$$

$$= 1\cdot\frac{1}{x} = \frac{1}{x}$$

By D' Alembert's Test

(I) if $\frac{1}{x} > 1$ or $x < 1$ ~~Then~~ i.e $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} > 1$

Then the given series Σu_n is convergent.

(II) if $\frac{1}{x} < 1$ or $x > 1$ i.e $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} < 1$

Then the given series Σu_n is divergent

~~Also~~ (III) if $\frac{1}{x} = 1$ Then $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1$

in this case D' Alembert's test fails.

When $x = 1$ Then $u_n = \frac{1}{n(n+1)}$

$$u_n = \frac{1}{n^2(1+1/n)} = \frac{1}{n^2}(1+1/n)^{-1}$$

$$u_n = \frac{1}{n^2}\left(1-\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}+\cdots\right)$$

To apply comparision test the Auxiliary series Σv_n whose nth term

$$v_n = \frac{1}{n^2}$$

Now $\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(1-\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}+\cdots\right)$

$= 1$ finite

Hence By Comparision test. $\Sigma v_n = 1/n^2$ being convergent. So the given series Σu_n is convergent

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∴ we conclude that the infinite series.

$$\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots\cdots$$

is convergent if $x < 1$ and
divergent if $x > 1$
convergent if $x = 1$

Ex Discuss the nature of the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Sol^n Given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

whose nth term $u_n = \sin\left(\frac{1}{n}\right)$

$$u_n = \left(\frac{1}{n} - \frac{1}{\lfloor 3}\cdot\frac{1}{n^3} + \frac{1}{\lfloor 5}\cdot\frac{1}{n^5} \cdots\cdots\right)$$

$$u_n = \frac{1}{n}\left(1 - \frac{1}{\lfloor 3}\cdot\frac{1}{n^2} + \frac{1}{\lfloor 5}\cdot\frac{1}{n^4} - \cdots\cdot\right)$$

Now nth term of Auxiliary series $\sum v_n$.

$$v_n = 1/n$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1 - \frac{1}{\lfloor 3}\cdot\frac{1}{n^2} + \frac{1}{\lfloor 5}\cdot\frac{1}{n^4} - \cdots\right)$$

$$= 1 \text{ finite}$$

Thus By Comparison test the series $\sum u_n$ is divergent because $\sum v_n = \sum 1/n$ is divergent.

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Q. Discuss the nature of series $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right) \cdot x^n$.

let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1} x^n$.

Whose nth term is

$$u_n = \frac{n}{n^2+1} \cdot x^n$$

$$u_{n+1} = \frac{n+1}{n^2+2} \cdot x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{n}{(n^2+1)} \times \frac{n^2+2}{(n+1)} \times \frac{x^n}{x^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{n} \times \cancel{n^2}(1+2/n^2)}{\cancel{n}(\cancel{n^2})(1+1/n)(1+\frac{1}{n^2})} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(1+2/n^2)}{(1+1/n)(1+1/n^2)} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

By D' Alembert's ratio test

(I) if $\frac{1}{x} > 1$ i.e $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$ i.e $x < 1$ Then the given series is convergent

(II) if $\frac{1}{x} \leq 1$ or $x > 1$ i.e $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} < 1$ Then the given series is divergent.

(III) if $\frac{1}{x} = 1$ or $x = 1$, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$ D' Alembert's test fails.

In this case $u_n = \frac{n}{n^2+1} = \frac{\cancel{n}}{n^2(1+1/n^2)} = \frac{1}{n}(1+1/n^2)^{-1}$

$$u_n = \frac{1}{n}\left(1 - 1/n^2 + \frac{1}{n^4} - \cdots\right)$$

$v_n = 1/n \implies \sum v_n$ is divergent

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(1 - \frac{1}{n^2} + \frac{1}{n^4} - \cdots\right) = 1 \text{ finite}$$

Hence By Comparison test given Series $\sum u_n$ for $x=1$ is divergent.